LIMITS

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$$\lim_{x \to a} f(x) = l$$

It would mean that when we approach the pt $x=a$ from the values which are just greater than or smaller than $x=a$, $f(x)$ will tend to move closer to the value l.

$$x=a, \quad \underbrace{\text{nbh}}_{\text{(neighbourhood)}} = (a-h, a+h)$$

It is eq. to say

$$(\forall x \in D(f))(\forall \epsilon > 0)(\exists\ \delta \in \mathbb{R}^+) \quad 0 < |x-a| < \delta \Rightarrow 0 < |f(x)-l| < \epsilon$$

- Basic statement about existence of limit :

If $f(x)$ is well-defined in the nbd($x=a$) & not necessarily at the pt. $x=a$, then we say that limit may exist, but if $f(x)$ is well defined at the pt $x=a$ & not in the nbd($x=a$), then we say limit does not exist.

eg. $\lim_{x \to \frac{\pi}{2}} \sec^{-1}(\sin x)$ does not exist

since $x \in \left(\frac{\pi}{2}-\delta, \frac{\pi}{2}+\delta\right) \Rightarrow |\sin x| \le 1$

$\Downarrow$

$\sec^{-1}(\sin x)$ does not exist

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→ **Left & Right Hand Limit**

$$LHL = \lim_{h \to 0} f(a-h) = \lim_{x \to a^-} f(x)$$

$$RHL = \lim_{h \to 0} f(a+h) = \lim_{x \to a^+} f(x)$$

- **Reasons for non-existence of limit**

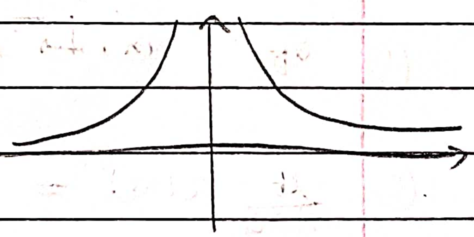
① LHL ≠ RHL
② Only one of LHL or RHL exists

- **One sided limit** – when only one side of limit about a pt exists, we take the limit of the fn at that pt to be the one sided limit

eg $\lim_{x \to 0} \sqrt{x} = 0$ (Here lim = RHL)

NOTE: Limit of a fn^n at a pt could be ∞

eg. $\lim_{x \to 0} \frac{1}{|x|} = \infty$



ALGEBRA OF LIMITS

Let $\lim_{x\to a} f(x) = L_1$ & $\lim_{x\to a} g(x) = L_2$ and both L_1 & L_2 are finite, then

(1) $\lim_{x\to a} c\,f(x) = c\left(\lim_{x\to a} f(x)\right) = cL_1$; $c \to$ const.

(2) $\lim_{x\to a} f(x) \pm g(x) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right) = L_1 \pm L_2$

(3) $\lim_{x\to a} f(x)\cdot g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a} g(x)\right)} = \dfrac{L_1}{L_2}$, $L_2 \neq 0$

eg. $\lim_{x\to 0} \dfrac{\sin x}{x} \neq \left(\underset{x\to 0}{\mathrm{Lt}} \sin x\right)\underbrace{\left(\underset{x\to 0}{\mathrm{Lt}} \dfrac{1}{x}\right)}_{\text{not finite}}$

FORMULAE

(1) If $p(x) \to$ polynomial, $\underset{x\to a}{\mathrm{Lt}}\ p(x) = p(a)$

(2) $\underset{x\to 0}{\mathrm{Lt}}\ (1+x)^{1/x} = e$

(3) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{e^x - 1}{x} = 1$

(4) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{(1+x)^n - 1}{x} = n$

(5) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\ln(1+x)}{x} = 1$

(6) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\sin^{-1}(x)}{x} = \underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\tan^{-1}(x)}{x} = 1$

(7) $\underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\sin x}{x} = \underset{x\to 0}{\mathrm{Lt}}\ \dfrac{\tan x}{x}$
$= \underset{x\to 0}{\mathrm{Lt}}\ \cos x = 1$

(8) $\lim\limits_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$ (9) $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln(a) \quad a \in \mathbb{R}^+$

(10) $\lim\limits_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{(m-n)}$ (11) $\lim\limits_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a(e), \quad a>0, \; a \neq 1$

If $\lim\limits_{x \to a} f(x) = 0$, then

(1) $\lim\limits_{x \to a} \frac{\sin f(x)}{f(x)} = \lim\limits_{x \to a} \frac{\tan f(x)}{f(x)} = \lim\limits_{x \to a} \cos f(x) = 1$

(2) $\lim\limits_{x \to a} \frac{\sin^{-1} f(x)}{f(x)} = \lim\limits_{x \to a} \frac{\tan^{-1} f(x)}{f(x)} = 1$

(3) $\lim\limits_{x \to a} \frac{b^{f(x)} - 1}{f(x)} = \ln(b)$; $b > 0$

(4) $\lim\limits_{x \to a} (1+f(x))^{1/f(x)} = e$

→ Series Expansions (about $x=0$)

(1) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

(2) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

(3) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} - \cdots$

(4) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(5) $a^x = 1 + \ell(a)x + \frac{\ell^2(a)x^2}{2!} + \ldots, \quad a \in \mathbb{R}^+$

(6) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots,$

$n \in \mathbb{R}, \quad |x| < 1$

(7) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots, \quad \underline{x \in (-1, 1]}$

→ Limit of composite fxn

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

$$\Uparrow$$

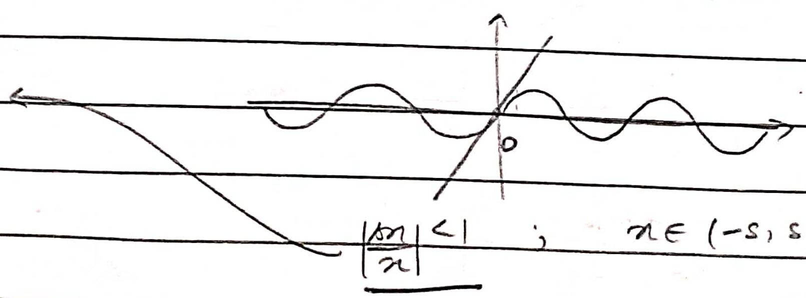
$f(x)$ is continuous at $x = \lim_{x \to a} g(x)$

eg. $\lim_{x \to 0} \left[\frac{\sin x}{x}\right]$, $[\cdot] \to$ GIF

$\neq \left[\lim_{x \to 0} \frac{\sin x}{x}\right] = 1$ because at $\lim_{x \to 0} [x] = 1$, $[x]$ is discontinuous

RHL : 0

LHL : 0



$\left|\frac{\sin x}{x}\right| < 1$; $x \in (-\delta, \delta)$

∵ LHL = RHL

∴ $\lim_{x \to 0} \left[\frac{\sin x}{x}\right] = 0$

Q ① $\lim_{x\to 1} [\sin^{-1}x] = 1$

$\lim_{x\to 1} \sin^{-1}x = \pi/2$ Graph discont @ $\pi/2$

LHL = $[\pi/2] = 1$

RHL → does not exist → 1

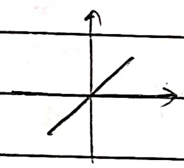
② $\lim_{x\to 0} \left[\frac{x}{\sin x}\right]$ LHL: 1, RHL: 1 → 1

③ $\lim_{x\to \infty} [\tan^{-1}(x)] = 1$ $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2 \to [\pi/2] = 1$

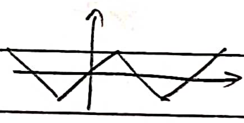
④ $\lim_{x\to -\infty} [\tan^{-1}(x)] = -2$ $\lim_{x\to -\infty} \tan^{-1}(x) = -\pi/2 \to [-\pi/2] = -2$

⑤ $\lim_{x\to 0} \left[\frac{\tan x}{x}\right]$ LHL: 1, RHL: 1 → 1

⑥ $\lim_{x\to 1} [\sin(\sin^{-1}x)] = 0$



⑦ $\lim_{x\to \pi/2} [\sin^{-1}(\sin x)] = 1$



INDETERMINATE FORMS

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^{\infty}, 0^0, \infty\cdot 0, \infty^0$

Q Evaluate

① $\lim_{x\to 1} \frac{\sqrt{1-\cos 2(x-1)}}{(x-1)}$

② $\lim_{x\to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$

A1. $\lim_{x\to 1} \frac{\sqrt{2}\,|\sin(x-1)|}{(x-1)}$

LHL: $-\sqrt{2}$, RHL: $\sqrt{2}$ → Does not exist.

A2. RHL: $\lim_{x\to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{e^{1/x}\left(1-\frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1+\frac{1}{e^{1/x}}\right)} = 1$

LHL: $\lim_{x\to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1} = -1$

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(3) $\lim_{x\to 1} \dfrac{x^3 - x^2\ell(x) + \ell(x) - 1}{(x-1)}$ (4) $\lim_{x\to 1} \dfrac{x^{1/3} + x^{1/2} + x^{3/2} - 3}{x^3 - 1}$

(5) $\lim_{n\to\infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

A3. $\dfrac{(x^3-1) - \ell(x)(x^2-1)}{(x-1)} = x^2+x+1 - \ell(x)(x+1)$

RHL: $\lim_{x\to 1^+} (\ldots) = 3 - (0) = 3$

LHL: $\lim_{x\to 1^-} (\ldots) = 3 - (0) = 3$

$\Rightarrow 3$

A4. $\dfrac{\frac{1}{3}(x^{-2/3}) + \frac{1}{2}(x^{-1/2}) + \frac{3}{2}(x^{1/2})}{3x^2} = \dfrac{\frac{1}{3}+\frac{1}{2}+\frac{3}{2}}{3} = \dfrac{7}{9}$

A5. $\dfrac{(n+1)!\,(n+3)}{(n+1)!\,(n+1)} = \dfrac{1}{1} = 1$

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(6) $\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - \sqrt{x^2+1}\right)$ (7) $\lim_{x\to\infty} \left(\dfrac{x^2+1}{x+1} - ax - b\right) = 0$

find a & b

(8) $\lim_{x\to -\infty} \left(\sqrt{x^2-8x} + x\right)$ (9.2) $\lim_{h\to 0} \dfrac{\ell(1+2h) - 2\ell(1+h)}{h^2}$

(10) $\lim_{x\to 0} \dfrac{(ab)^x - a^x - b^x + 1}{x^2}$ (11) $\lim_{x\to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x}$

(12) $\lim_{x\to 0} \dfrac{ae^x - b}{x} = 2$, find a & b. (9.1) $\lim_{x\to 0} \dfrac{s(\pi\cos^2 x)}{x^2}$

A6. $\dfrac{x}{\sqrt{x^2+x+1}+\sqrt{x^2+1}} = \dfrac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x^2}}} = 1/2$

A7. $\dfrac{x^2+1-ax^2-ax-bx-b}{x+1} = \dfrac{\overset{0}{\boxed{(1-a)}}x^2-(a+b)x-(b+1)}{(x+1)}$ (for finite lt.)

$a=1 \Rightarrow -\left[\dfrac{(a+b)x+(b+1)}{(x+1)}\right] = -\left(\dfrac{a+b+\cancel{\frac{b+1}{x}}}{1+\cancel{\frac{1}{x}}}\right)$

$(a,b)=(1,-1)$ $\quad = -(a+b)=0$

$\Rightarrow b=-1$

A8. $\dfrac{8x}{x-\sqrt{x^2-8x}} \to \dfrac{8}{1-\underbrace{\sqrt{1-\frac{8}{x}}}_{1^+}} \longrightarrow \infty$ ✗

$\dfrac{8x}{x-\sqrt{x^2(1-\frac{8}{x})}} = \dfrac{8x}{x-|x|\sqrt{1-\frac{8}{x}}}$

$x\to-\infty$ $\quad = \dfrac{8\cancel{x}}{\cancel{x}\left(1+\sqrt{1-\frac{8}{x}}\right)}$

$\Rightarrow |x|=-x$

$= 4$

A9.2 $\dfrac{2h-\frac{(2h)^2}{2}-2\left(h-\frac{h^2}{2}\right)}{h^2} = -1$

A10. $\left(\dfrac{a^x-1}{x}\right)\left(\dfrac{b^x-1}{x}\right) = \ell(a)\,\ell(b)$

A11. $\dfrac{(1+\tan x)-(1+x)}{\tan x-x} = 1$

A12. $\dfrac{a+ax-b}{x} \Rightarrow \dfrac{ax+(a-b)}{x}$

$a=2 \Rightarrow (a,b)=(2,2)$

$a=b \Rightarrow b=2$

A9.1 $\dfrac{\sin(\pi(1-\sin^2x))}{x^2} = \dfrac{\sin(\pi\sin^2x)}{\pi(\sin^2x)}\left(\dfrac{\sin^2x}{x^2}\right)\pi = \pi$

NOTE: $\lim_{x \to \infty} a^x = \begin{cases} \infty, & a > 1 \\ 1, & a = 1 \\ 0, & a \in (0,1) \end{cases}$

$\longrightarrow \lim_{x \to a} (f(x))^{g(x)}$

Existence: $f(x) > 0$ in nbd $(x = a)$

$$\lim_{x \to a} (f(x))^{g(x)} = \begin{cases} e^{\lim_{x \to a} g(x)[f(x)-1]}, & \lim_{x \to a} f(x) = 1 \\ e^{\lim_{x \to a} g(x) \ln(f(x))}, & \lim_{x \to a} f(x) \neq 1 \end{cases}$$

Proof: $y = f(x)^{g(x)} \Rightarrow y = e^{g(x)\, l(f(x))}$

$\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x)\, l(f(x))}$

$= \boxed{e^{\lim_{x \to a} g(x)\, l(f(x))}}$

Since $\lim_{x \to a} p(q(x)) = p\left(\lim_{x \to a} q(x)\right)$

given $p(x)$ continuous at $\lim_{x \to a} q(x)$

$p \to e^x$

$q \to g(x)\, l(f(x))$

if $\lim_{x \to a} f(x) = 1 \Rightarrow \lim_{x \to 0} l(f(x)) = 0$

$\Rightarrow l(1 + (f(x)-1)) = f(x) - 1$

(series exp of $l(1+x)$ @ $x = 0$)

$\Rightarrow \lim_{x \to a} f(x)^{g(x)} = e^{\lim_{x \to a} g(x)[f(x)-1]}$

$\left\{ \lim_{x\to 0} (x)^{\sin x} \right.$ does ~~not~~ exist. since $x>0$ in nbd$(x\to 0)$ $\left.\right\}$

eg. (i) $\lim_{x\to 0} (|x|)^{\sin x} = e^{\lim_{x\to 0} \sin x \, l(|x|)} = \boxed{1}$

$\lim_{x\to 0} \sin x \, l(|x|) = \lim_{x\to 0} \frac{l(|x|)}{\operatorname{cosec}(x)} = \lim_{x\to 0} \frac{(1/x)}{-\operatorname{cosec}\cot}$ (L'Hôpital)

$= \lim_{x\to 0} - \frac{s_x^2}{x^2 c_x} x \to 0$

(13) $\lim_{x\to 0} \left(t\left(\frac{\pi}{4}+x\right)\right)^{1/x}$

(14) $\lim_{x\to \infty} \left(\frac{x+6}{x+1}\right)^{(x+4)}$

(15) $\lim_{x\to 0} \left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}$

(16) $\lim_{x\to \infty} \left(\frac{x-3}{x+2}\right)^{x}$

A13 1. Exists.

2. $e^{\lim_{x\to 0} \frac{1}{x} l\left(t\left(\frac{\pi}{4}+x\right)\right)}$

$\lim_{x\to 0} \frac{1}{x}\left(t\left(\frac{\pi}{4}+x\right)-1\right) = \lim_{x\to 0} \frac{1}{x}\left(\frac{1+t}{1-t}-1\right) = \lim_{x\to 0} \frac{2 t_x}{x(1-t_x)}$

$= \boxed{2}$

$\Rightarrow \boxed{e^2}$

A14. 1. Exists.

2. $e^{\lim_{x\to \infty} (x+4) \, l\left(\frac{x+6}{x+1}\right)} = e^{\lim_{x\to \infty} (x+4)\left(\frac{5}{x+1}\right)} = e^{\lim_{x\to \infty} 5\left(1+\frac{3}{x+1}\right)}$

$= \boxed{e^5}$

A15. 1. Exists

2. $e^{\lim_{x\to 0} \frac{1}{x^2} l\left(\frac{1+5x^2}{1+3x^2}\right)} = e^{\lim_{x\to 0} \left(\frac{1}{\cancel{x^2}}\right)\left(\frac{2\cancel{x^2}}{1+3x^2}\right)} = \boxed{e^2}$

A16. 1. Exists

2. $e^{\lim_{x\to \infty} x \, l\left(\frac{x-3}{x+2}\right)} = e^{\lim_{x\to \infty} x\left(\frac{x-3-x-2}{x+2}\right)} = e^{\lim_{x\to \infty} -5\left(\frac{1}{1+\frac{2}{x}}\right)} = \boxed{e^{-5}}$

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L'Hôpital's Rule

Consider

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$$

Here, $f(x)$ & $g(x)$ must be differentiable in the nbd $(x=a)$

Then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\Rightarrow \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$

$\vdots$

Q.

A. (i) $\lim_{x\to 1} \frac{\sqrt{f(x)}-1}{\sqrt{x}-1} \overset{(LH)}{=} \frac{f'(x)}{\cancel{2}\sqrt{f(x)}} \times \cancel{2}\sqrt{x} = \frac{2(1)}{\sqrt{1}}$

$= 2$

(ii) $\lim_{x\to 2} \frac{4x-2f(x)}{x-2} \overset{(LH)}{=} \frac{4-2f'(x)}{1} = 4-2(4) = -4$

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(iii) $\lim_{x\to a} \dfrac{kg(x) - k - kf(x) + k}{g(x) - f(x)} \overset{(LH)}{=} \dfrac{kg'(x) - kf'(x)}{g'(x) - f'(x)} = k$

$\Rightarrow \boxed{k = 4}$

(iv) $\lim_{h\to 0} \dfrac{(2h+2)\, f'(h^2+2h+2)}{(1-2h)\, f'(h-h^2+1)} \overset{(LH)}{=} \left(\dfrac{2}{1}\right) \dfrac{f'(2)}{f'(1)} = 3$

Q (i) If $f(1) = 1$, $f'(1) = 2$, find $\lim_{x\to 1} \dfrac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$

(ii) If $f(2) = 4$, $f'(2) = 4$, then find

$\lim_{x\to 2} \dfrac{xf(2) - 2f(x)}{x - 2}$

(iii) Let $f(a) = g(a) = k$, if $\lim_{x\to a} \dfrac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)}$

$= 4$

then find k

(iv) Given that $f'(2) = 6$ and $f'(1) = 4$,

find $\lim_{h\to 0} \dfrac{f(h^2+2h+2) - f(2)}{f(h-h^2+1) - f(1)}$

Sandwich Theorem

If $\forall x$ in nbd$(x=a)$,

$$h(x) \le f(x) \le g(x)$$

$$\Rightarrow \lim_{x\to a} h(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

if $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = k$

$$\Rightarrow \boxed{\lim_{x\to a} f(x) = k}$$

eg - $\lim_{x\to\infty} \frac{l(x) - [x]}{[x]} = \left(\lim_{x\to\infty} \frac{l(x)}{[x]}\right) - 1$

Since, $(x-1) < [x] \le x$

$$\Rightarrow \frac{1}{x} \le \frac{1}{[x]} < \frac{1}{(x-1)}$$

$$\Rightarrow \frac{l(x)}{x} \le \frac{l(x)}{[x]} < \frac{l(x)}{(x-1)}$$

$$\Rightarrow \underbrace{\lim_{x\to\infty} \frac{l(x)}{x}}_{1/x} \le \lim_{x\to\infty} \frac{l(x)}{[x]} < \underbrace{\lim_{x\to\infty} \frac{l(x)}{(x-1)}}_{1/x}$$

(LH) (LH)

$= 0$ $= 0$

$$\Rightarrow \lim_{x\to\infty} \frac{l(x)}{[x]} = 0 \Rightarrow \lim_{x\to\infty} \frac{l(x) - [x]}{[x]} = -1$$

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Q. $\lim_{n\to\infty} \dfrac{[x]+[2x]+\dots+[nx]}{n}$

A. $kx-1 < [kx] \le kx \Rightarrow \dfrac{n^2x+nx-2n}{2n^2} \le \dfrac{\sum [kx]}{n^2} \le \dfrac{n(n+1)x}{2n^2}$

$$\lim_{n\to\infty} \underbrace{\frac{n^2x+nx-2n}{2n^2}}_{\frac{1}{2}\left(x+\cancel{\frac{x-2}{n}}\right)} \le (\dots) \le \lim_{n\to\infty} \underbrace{\frac{n^2x+nx}{2n^2}}_{\frac{1}{2}\left(x+\cancel{\frac{x}{n}}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sum [kx]}{n^2} = \frac{x}{2}$$

LEIBNITZ RULE

Let $F(x) = \int_{l(x)}^{u(x)} f(x)\, dx$

$$\frac{dF(x)}{dx} = f(u(x))\, u'(x) - f(l(x))\, l'(x)$$

Q. Evaluate

1. $\lim_{x\to 0} \left(\frac{1}{x^5} \int_0^x e^{-t^2} dt - \frac{1}{x^4} + \frac{1}{3x^2} \right)$

2. If $\lim_{x\to 0} \left(\dfrac{\int_0^{x^2} \sin^2 t\, dt}{x^n} \right)$ is a non-zero finite no, find (n)

$1 - x^2 + \frac{x^4}{2}$

A. 1. $\left(\frac{0}{0}\right) \Rightarrow (LH)$

$$\lim_{x\to 0} \frac{\boxed{e^{-x^2}} - 1 + \frac{\cancel{2}x^2}{\cancel{2}}}{5x^4}$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{10} = \frac{1}{10}$$

2. $(0/0) \Rightarrow (LH)$

$$\lim_{x\to 0} \frac{(\sin x^4)(2x)}{n x^{(n-1)}} = \lim_{x\to 0} \left(\frac{2}{n}\right)\left(\frac{\sin x^4}{x^4}\right) \frac{1}{x^{(n-6)}}$$

$$n - 6 = 0 \Rightarrow \boxed{n = 6}$$

CONTINUITY

& DIFFERENTIABILITY

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CONTINUITY

- Continuity – A f^n $y = f(x)$ is said to be continuous at $x = a$, if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

i.e LHL = RHL = $f(a)$

OR

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

NOTE: A f^n $y = f(x)$ will be discont. at $x = a$ in any of the following cases :–

1. LHL & RHL exist, but are not equal.
2. LHL & RHL exist & are equal but not equal to $f(a)$.
3. $f(a)$ is not defined
4. At least one of the limits does not exist.

→ **Ppts of cont. f^n**

Let $f(x)$ & $g(x)$ both be cont. at $x=a$, then

① $c\,f(x)$ is cont. at $x=a$

② $af(x) \pm bg(x)$ is cont. at $x=a$

③ $f(x)\cdot g(x)$ is cont. at $x=a$

④ $\dfrac{f(x)}{g(x)}$ is cont. at $x=a$ provided $g(a) \neq 0$

→ **Continuity in an interval**

(I) Open interval – $f(x)$ is said ~~to~~ be cont. in (a, b) if it is cont at every pt. in the interval.

(II) Closed interval – $f(x)$ is cont. in $[a, b]$ if

II.I $f(x)$ is cont. in (a, b)

II.II $\lim\limits_{x \to a^+} f(x) = f(a)$

II.III $\lim\limits_{x \to b^-} f(x) = f(b)$

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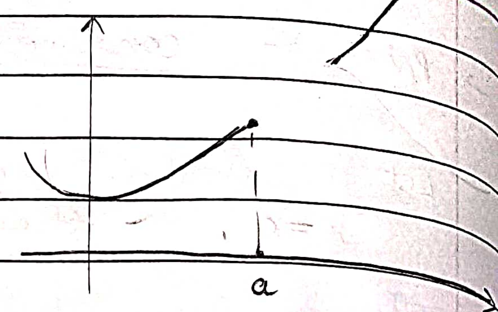
NOTE: Here, only left neighbourhood is defined.



& $\lim_{x \to a^-} f(x) = f(a)$

so, $f(x)$ is cont. at $x = a$

→ Continuity of Composite fxn

For checking continuity of $y = f(g(x))$, check continuity at controversial pts. of $f(g(x))$, $g(x)$ & $f(x)$

Q. Find pts. of discont. of $y = \frac{1}{u^2+u-2}$ where $u = \frac{1}{(x-1)}$

A. ① $\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)}$ $\quad x = 1, 2$

② $\frac{1}{(x-1)}$ $\quad x = 1$

③ $\frac{1}{\left(\frac{1}{x-1}+2\right)\left(\frac{1}{x-1}-1\right)} = \frac{-(x-1)}{(2x-1)(x-2)}$ $\quad x = 1, 2, 1/2$

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→ Removable & Non-removable Discontinuity

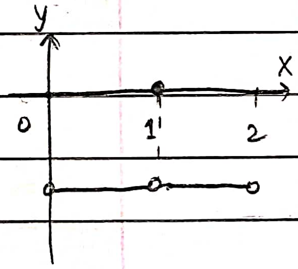
$$f(x) = [x] + [-x], \quad x \in (0,2)$$

$f(x)$ is discont. at $x = 1$, since $\lim_{x \to 1} f(x) = -1$

but $f(1) = 0$

If we redefine $f(x)$ as follows,

$$f(x) = \begin{cases} [x] + [-x], & x \in (0,1) \cup (1,2) \\ -1, & x = 1 \end{cases}$$



the f^n becomes cont.

Hence, $f(x)$ is said to have a removable discont. at $x = 1$.

In general, if $\lim_{x \to a} f(x)$ exists

& $f(x)$ is discont. at $x = a$,

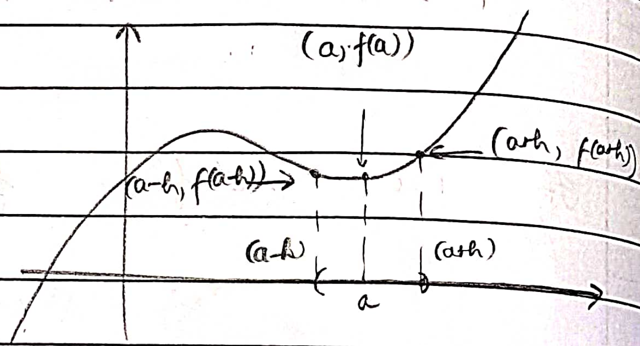
then $x = a$ is a removable discont.

DIFFERENTIABILITY

Def: A fxn $y = f(x)$ is said to be differentiable at $x=a$, if LHD & RHD exist & are equal & FINITE.

LHD: $\lim_{h \to 0^+} \dfrac{f(a)-f(a-h)}{a-(a-h)}$

$= \lim_{h \to 0^+} \dfrac{f(a)-f(a-h)}{h}$



RHD: $\lim_{h \to 0^+} \dfrac{f(a+h)-f(a)}{(a+h)-a} = \lim_{h \to 0^+} \dfrac{f(a+h)-f(a)}{h}$

NOTE: Fxn $y = f(x)$ is said to be non-diff at $x=a$ if :-

① $f'(a^+)$ & $f'(a^-)$ exist but are not equal

② Either or both $f'(a^+)$ & $f'(a^-)$ are not finite.

③ Either or both $f'(a^+)$ & $f'(a^-)$ do not exist

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NOTE: (1) If $f(x)$ is diff at $x=a$, then it must be cont. but converse is not true.

eg – $y = |x|$ – Cont. but not diff.

LHD $\lim_{h \to 0^+} \frac{|0| - |0-h|}{h} = \frac{-(h)}{h} = -1$

RHD $\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \frac{h}{h} = 1$

(2) Non cont. $\Rightarrow$ Non. diff.

☆(3) If LHD & RHD both are finite & not equal, then it is non-diff but ALWAYS cont.

REMARK: So if both cont & diff are to be checked, check diff first!

(4) Sharp Corner Thumb Rule –
If sharp pt. in graph, fxn GENERALLY, fxn non-diff at that pt.

(5) Diff at end pts of $[a, b]$

$f'(a^+)$ & $f'(b^-)$ must exist & be finite.

☆ (6) If $y = f(x)$ is diff at $x = a$, then it is not necessary that $f'(x)$ is cont. at $x = a$

eg -

$$f(x) = \begin{cases} x^2 s\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

LHD: $\lim\limits_{h \to 0^+} \dfrac{f(0) - f(0-h)}{h} = \dfrac{0 - (-h)^2 s\left(\frac{-1}{h}\right)}{h}$

$= \dfrac{s(1/h)}{(1/h) \to \infty} = 0$

RHD: $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \dfrac{h^2 s\left(\frac{1}{h}\right) - 0}{h}$

$= \dfrac{s(1/h)}{(1/h) \to \infty} = 0$

$\therefore$ $f(x)$ is diff at $x = 0$

However,

$$f'(x) = \begin{cases} x^2 s\left(\frac{1}{x}\right)\left[\frac{2x}{x^2} + \frac{c(1/x)}{s(1/x)}\left(\frac{-1}{x^2}\right)\right], & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 2x\, s\left(\frac{1}{x}\right) - c\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

<u>LHL:</u> $\lim_{h \to 0^+} 2(-h)\, s\left(\frac{-1}{h}\right) - c\left(\frac{-1}{h}\right) = \underbrace{\frac{2 s(1/h)}{(1/h)}}_{0} - \underbrace{c(1/h)}_{\text{(cannot comment on behavior)}}$

<u>RHL:</u> $\lim_{h \to 0^+} \underbrace{2h\, s\left(\frac{1}{h}\right)}_{0} - \underbrace{c\left(\frac{1}{h}\right)}_{\text{(cannot comment on behaviour)}}$

→ <u>Application of First Principle</u>

<u>eg.</u> Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$, $\forall\, x, y \in \mathbb{R}$

If $f'(0)$ exists & is equal to -1 & $f(0) = 1$, find $f(2)$

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A.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{2x+2h}{2}\right) - f(x)}{h}$$

(Given)

$$= \lim_{h\to 0} \frac{\frac{f(2x)+f(2h)}{2} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(2x) + f(2h) - 2f(x)}{2h}$$

if $y=0$, $\Rightarrow$ $f\left(\frac{x}{2}\right) = \frac{f(x)+f(0)}{2} = \frac{f(x)+1}{2}$

$x \to 2x$ $\Rightarrow$ $f(x) = \frac{f(2x)+1}{2}$

$\Rightarrow$ $\underline{f(2x) = 2f(x) - 1}$

$$= \frac{2f(x) - 1 + f(2h) - 2f(x)}{h}$$

$$= \frac{f(2h) - 1}{2h} = \frac{f(0+2h) - f(0)}{2h}$$

$$\Rightarrow f'(x) = \lim_{(2h)\to 0} \frac{f(0+(2h)) - f(0)}{(2h)}$$

$2h \to h$ $$= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$\Rightarrow f'(x) = -1$ $\Rightarrow$ $f(x) = \underline{c - x}$

since $f(0) = 1 \Rightarrow 1 = c - 0 \Rightarrow \underline{c = 1} \Rightarrow f(x) = 1 - x$

$\therefore$ $f(2) = 1 - 2 = -1$

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Q. 1. $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x} - \frac{2}{e^{2x}-1}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Find k s.t $f(x)$ is cont. at $x=0$.

A. LHL: $\lim_{h \to 0^+} \frac{1}{h} - \frac{2}{e^{2h}-1} = \frac{1}{h} - \frac{2}{1+2h+2h^2-1}$

$= \frac{1}{h} - \frac{1}{h+h^2} = \frac{1}{h}\left(\frac{h}{h+1}\right) = \frac{1}{h+1}$

$= 1$

RHL: $\lim_{h \to 0^+} \frac{-1}{h} - \frac{2}{e^{-2h}-1} = \frac{-1}{h} - \frac{2}{1-2h+2h^2-1}$

$= \frac{-1}{h} + \frac{1}{h-h^2} = \frac{1}{h}\left(\frac{h}{1-h}\right) = \frac{1}{(1-h)}$

$= 1$

$\Rightarrow k=1$

Q. $f(x) = \begin{cases} (x-1)\, s\left(\frac{1}{x-1}\right), & x \neq 1 \\ 0, & x = 1 \end{cases}$

Check diff. at $x = 0, 1$

A. ① $x=0$

R': $\lim_{h\to 0^+} \frac{(h-1)\, s\left(\frac{1}{h-1}\right) - s_1}{h}$

$\overset{(LH)}{=} s\left(\frac{1}{h-1}\right) - c\left(\frac{1}{h-1}\right)\left(\frac{1}{h-1}\right) = c_1 - s_1$

L': $\lim_{h\to 0^+} \frac{(-h-1)\, s\left(\frac{1}{-h-1}\right) - s_1}{(-h)} = \frac{s_1 - (h+1)\, s\left(\frac{1}{h+1}\right)}{h}$

$\overset{(LH)}{=} -\left[s\left(\frac{1}{h+1}\right) - c\left(\frac{1}{h+1}\right)\left(\frac{1}{h+1}\right)\right] = c_1 - s_1$

② $x=1$

R': $\lim_{h\to 0^+} \frac{(1+h-1)\, s\left(\frac{1}{1+h-1}\right) - 0}{h}$

$= s\left(\frac{1}{h}\right) \longrightarrow$ Not defined

L': $\lim_{h\to 0} \frac{(1-h-1)\, s\left(\frac{1}{1-h-1}\right) - 0}{-h}$

$= -s\left(\frac{1}{h}\right) \longrightarrow$ Not defined.

$\Rightarrow$ Non-diff at $x=1$

Diff at $x=0$

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Q. Let $f(x) = [x]\, c_{\left(\frac{2x-1}{2}\right)\pi}$. Test cont. of $f(x)$ at $x \in \mathbb{Z}$

A. R: $\lim_{h\to 0^+} [x+h]\, c_{\left(\frac{2x+2h-1}{2}\right)\pi} = x\, c_{\frac{\pi}{2} - \pi(x+h)}$
$= x\, s_{\pi(x+h)}$
$= x\, s_{\pi x} = \underline{0} \quad (x \in \mathbb{Z})$

L: $\lim_{h\to 0^+} [x-h]\, c_{\left(\frac{2x-2h-1}{2}\right)\pi} = (x-1)\, c_{\left(\frac{\pi}{2} - (x-h)\pi\right)}$
$= (x-1)\, s_{\pi(x-h)}$
$= (x-1)\, s_{\pi x} = \underline{0} \quad (x \in \mathbb{Z})$

x: $c_{\frac{\pi}{2} - \pi x} = s_{\pi x} = 0 \quad (x \in \mathbb{Z})$

$L = R = f(x) \Rightarrow$ Continuous

Q. $f(x) = [x]^2 - [x^2]$ Check continuity of $f(x)$ at $x \in \mathbb{Z}$

A. R: $\lim_{h\to 0^+} [x+h]^2 - [(x+h)^2] = x^2 - [x^2 + 2hx + h^2]$
$= x^2 - x^2 = 0$

L: $\lim_{h\to 0^+} [x-h]^2 - [(x-h)^2] = (x-1)^2 - [x^2 - 2hx + h^2]$
$= (x^2 - 2x + 1) - (x^2 - 1)$
$= \underline{2 - 2x}$

x: $x^2 - x^2 = 0$

For continuity $L = R = f(x) \Rightarrow 0 = 2 - 2x \Rightarrow x = 1$
$\Rightarrow$ Cont. only at $\underline{x = 1}$

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Q. $f(x) = (x^2-1)\,|x^2-3x+2| + \cos|x|$

Find pts. at which $f(x)$ is non-diff.

A. Possible pts. $x = 0, 1, 2$

$$f(x) = \begin{cases} (x+1)(x-1)^2(x-2) + \cos x, & x \geq 2 \\ (x+1)(x-1)^2(2-x) + \cos x, & x \in [1,2) \\ (x+1)(x-1)^2(x-2) + \cos x, & x \in [0,1) \\ (x+1)(x-1)^2(x-2) + \cos x, & x < 0 \end{cases}$$

$x = 0, 1$ need not be checked

since

$x=2$

$$R': \lim_{h\to 0^+} \frac{(h+3)(h+1)^2(h) + \cos(2+h) - \cos 2}{h}$$

$$= (h+3)(h+1)^2 + \frac{2\sin\left(\frac{h}{2}\right)\sin\left(\frac{h+4}{2}\right)}{h}$$

$$= 3 + \sin 2$$

$$L': \lim_{h\to 0^+} \frac{(3-h)(1-h)^2(-h) + \cos(2-h) - \cos 2}{(-h)}$$

$$= (3-h)(1-h)^2 + \frac{2\sin\left(-\frac{h}{2}\right)\sin\left(\frac{4-h}{2}\right)}{h}$$

$$= 3 - \sin 2$$

$R' \neq L' \Rightarrow f(x)$ is non-diff at $x = 2$

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Q. $f(x) = [x]\sin\pi x$. Find $f'(k^-)$.

A. $f'(k^-) = \lim_{h\to 0^+} \dfrac{[k-h]\sin\pi(k-h) - [k]\sin\pi k}{h}$

$= \dfrac{(k-1)\sin(\pi k - \pi h) - k\sin\pi k}{h}$

$= \dfrac{2k\sin\left(\frac{-\pi h}{2}\right)\cos\left(\pi k - \frac{\pi}{2}h\right)}{h} - \dfrac{\sin(\pi k - \pi h)}{h}$

$(k \in \text{even})$ $= -\pi k\cos(\pi k) + \dfrac{\sin\pi h}{h}$

$= -\pi k(1) + \pi = \underline{\pi(1-k)}$

$(k \in \text{odd})$ $= -\pi k(-1) - \dfrac{\sin\pi h}{h} = \underline{-\pi(1+k)}$

Q. $f(x) = \dfrac{x}{1+|x|}$. Find pts. at which x is non-diff.

A. Possible pts. $\underline{x = 0}$.

$\left.\begin{aligned} &\underline{R'}: \lim_{h\to 0^+} \dfrac{\frac{h}{1+h} - 0}{h} = 1 \\ &\underline{L'}: \lim_{h\to 0^+} \dfrac{\frac{-h}{1+h} - 0}{-h} = 1 \end{aligned}\right\} \Rightarrow$ Diff $\forall\ \underline{x \in R}$

Q. Let $f(x)$ be cont. fx^n, defined in $[1, 3]$.
If $f(x)$ takes only rational values $\forall x$ & $f(2) = 10$.
Then find $f(1.5)$.

A. Since $\exists$ infinitely many irrational nos. b/w any 2 rational nos.,
$f(x)$ cannot take any value except 10.

$\Rightarrow \underline{f(x) = 10} \quad \forall x \in [1, 3]$

$\Rightarrow \underline{f(1.5) = 10}$

Q. For every integer 'n', let a_n & $b_n \in \mathbb{R}$.
Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} a_n + \sin \pi x \; ; & x \in [2n, 2n+1] \\ b_n + \cos \pi x \; ; & x \in (2n-1, 2n) \end{cases}$$

If $f(x)$ is cont., then find

a) $a_n - b_n$
b) $a_{(n-1)} - b_n$

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A a) $x = 2n$,

$L: \lim_{h \to 0^+} b_n + c(2n\pi - \pi h)$

$= 1 + b_n$

$R: \lim_{h \to 0^+} a_n + s(2n\pi + \pi h) = a_n$

$x: a_n$

$L = R = f(x) \Rightarrow 1 + b_n = a_n \Rightarrow a_n - b_n = 1$

b) $x = 2n - 1$,

$L: \lim_{h \to 0^+} a(n-1) + s(2n-1)\pi - \pi h$

$= a(n-1)$

$R: \lim_{h \to 0^+} b_n + c(2n-1)\pi + \pi h$

$= b_n - 1$

$x: a(n-1)$

$L = R = f(x) \Rightarrow a(n-1) = b_n - 1 \Rightarrow a(n-1) - b_n = 1$

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Q. If $f''(x) = -f(x)$ & $g(x) = f'(x)$

If $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and $F(5)=5$,

find $F(10)$

A. $F'(x) = f\left(\frac{x}{2}\right) f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right) g'\left(\frac{x}{2}\right)$

$= f\left(\frac{x}{2}\right) f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right) \underbrace{f''\left(\frac{x}{2}\right)}_{-f(x/2)}$

$= 0$

$\Rightarrow F(x) = C \Rightarrow F(x) = 5$ (since $F(5)=5$)

$\Rightarrow \underline{F(10) = 5}$

Q. Let $f(x) = \begin{cases} -1, & x \in [-2,0] \\ x-1, & x \in (0,2] \end{cases}$

$g(x) = f(|x|) + |f(x)|$.

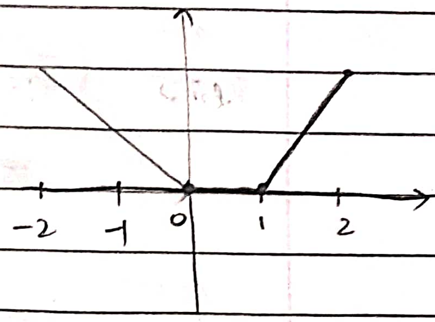
Test cont. & diff of $f(x)$ in $x \in [-2,2]$.

A $f(|x|) = |x| - 1$

$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases} = \begin{cases} -1, & -1 \geq 0, & x \in [-2,0] \\ x-1, & x-1 \geq 0, & \underline{x \in (0,2]} \\ & \Rightarrow \underline{x \geq 1} & \\ 1, & -1 < 0, & x \in [-2,0] \\ 1-x, & x-1 < 0, & x \in (0,2] \\ & \underline{x < 1} & \end{cases}$

$$= \begin{cases} 1, & x \in [-2, 0] \\ 1-x, & x \in (0, 1) \\ x-1, & x \in [1, 2] \end{cases}$$

$$g(x) = \begin{cases} -x, & x \in [-2, 0] \\ 0, & x \in (0, 1) \\ 2x-2, & x \in [1, 2] \end{cases}$$



$\Rightarrow$ Cont. $\forall\ x \in [-2, 2]$

Non-diff at $\underline{x = 0, 1}$

Q. If a fn^n $f: [-2a, 2a] \to \mathbb{R}$ is an odd fn^n & $f(x) = f(2a-x)$ $\forall\ x \in [a, 2a]$.

If LHD at $x = a$ is 0., find LHD at $x = -a$.

A. $$\lim_{h \to 0^+} \frac{f(-a-h) - f(-a)}{(-h)} = \frac{\overbrace{f(a+h)}^{f(2a-(a+h))} - f(a)}{(-h)} = \frac{f(a-h) - f(a)}{-h}$$

$$= -f'(a^-) = \underline{0}$$

Q. $f'(0) = \lim_{n \to \infty} n f\left(\frac{1}{n}\right)$ & $f(0) = 0$.

Using this, find $\lim_{n \to \infty} (n+1)\left(\frac{2}{\pi}\right) c^{-1}\left(\frac{1}{n}\right) - n$

$; \left|c^{-1}\left(\frac{1}{n}\right)\right| < \frac{\pi}{2}$

$h = 1/n$

A. $f'(0) = \underset{h \to 0^+}{lt} \dfrac{f(h) - \overset{0}{\overline{f(0)}}}{h} = \underset{n \to \infty}{lt} \; n f\left(\frac{1}{n}\right)$

So, $\underset{h \to 0^+}{lt} \left(\frac{1}{h} - 1\right)\left(\frac{2}{\pi}\right) c^{-1}_{h} - \frac{1}{h} = \underset{n \to \infty}{lt} (n-1)\left(\frac{2}{\pi}\right) c^{-1}_{\left(\frac{1}{n}\right)} - n$

$$= \frac{\left(\frac{2}{\pi}\right)(1-h)\, c^{-1}_{h} - 1}{h}$$

(LH)

$$= \left(\frac{2}{\pi}\right)\left[\frac{h-1}{\sqrt{1-h^2}} - c^{-1}_{h}\right] = \left(1 - \frac{2}{\pi}\right)$$

Q. If $|c| < 1/2$ & $f(x)$ is diff at $x = 0$ given by

$$f(x) = \begin{cases} b\, s^{-1}\left(\frac{c+x}{2}\right), & x \in (-1/2, 0) \\ \frac{1}{2}, & x = 0 \\ \dfrac{e^{ax/2} - 1}{x}, & x \in (0, 1/2) \end{cases}$$

Find the value of 'a' & prove that $64b^2 = 4 - c^2$

$\left(1+\frac{ah}{2}+\frac{a^2h^2}{8}\right)$

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A. R':

$$\lim_{h\to 0^+} \frac{\frac{e^{ah/2}-1}{h}-1/2}{h} = \frac{2e^{ah/2}-2-h}{2h^2}$$

$$= \frac{(a-1)}{2h} + \frac{a^2}{8}$$

L':

$$\lim_{h\to 0^+} \frac{b\sin^{-1}\left(\frac{c-h}{2}\right)-\frac{1}{2}}{-h} \overset{(LH)}{=} \frac{+b}{2\sqrt{1-\left(\frac{c-h}{2}\right)^2}} = \frac{b}{\sqrt{4-c^2}}$$

if $b\sin^{-1}\left(\frac{c}{2}\right)=\frac{1}{2}$

For diff., $a=1 \Rightarrow \frac{b}{\sqrt{4-c^2}} = \frac{1}{8}$

$\Rightarrow 64b^2 = 4-c^2$

Q Let $F(x) = f(x)\,g(x)\,h(x)$ $\forall x\in\mathbb{R}$, f, g, h are diff.

At some pt. x_0

$F'(x_0) = 21F(x_0)$

$f'(x_0) = 4f(x_0)$

$g'(x_0) = -7g(x_0)$

$h'(x_0) = k\,h(x_0)$

Find k.

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A $F'(x_0) = f(x_0)\,g(x_0)\,h(x_0)\left[\dfrac{f'(x_0)}{f(x)} + \dfrac{g'(x_0)}{g(x_0)} + \dfrac{h'(x_0)}{h(x_0)}\right]$

$21\,F(x_0) = (k-3)\;F(x_0)$

$\Rightarrow \underline{k = 24}$

☆ **Q** Let $g(x) = \ell(f(x))$

$f(x)$ is a twice diff. +ve fn^n on $(0,\infty)$ s.t

$$f(x+1) = x\,f(x)$$

Then, for $n \in \mathbb{N}$, prove that

$$g''\left(n+\tfrac{1}{2}\right) - g''\left(\tfrac{1}{2}\right) = -4\left[1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2n-1)^2}\right]$$

A. $g(x+1) - g(x) = \ell\left(\dfrac{f(x+1)}{f(x)}\right) = \ell(x)$

$\Rightarrow g'(x+1) - g'(x) = \dfrac{1}{x}$

$\Rightarrow g''(x+1) - g''(x) = -\dfrac{1}{x^2}$

$x = \frac{1}{2}$ $\quad g''\left(\frac{3}{2}\right) - g''\left(\frac{1}{2}\right) = \dfrac{-4}{1^2}$

$x = \frac{3}{2}$ $\quad g''\left(\frac{5}{2}\right) - g''\left(\frac{3}{2}\right) = \dfrac{-4}{3^2}$

$\vdots$

$x = \frac{(2n-1)}{2}$ $\quad g''\left(n+\frac{1}{2}\right) - g''\left(n-\frac{1}{2}\right) = \dfrac{-4}{(2n-1)^2}$

$g''\left(n+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left(\Sigma \dfrac{1}{(2n-1)^2}\right)$

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Q. Let $f: \mathbb{R} \to \mathbb{R}$ s.t $|f(x)-f(y)| \leq |x-y|^3 \quad \forall x,y \in \mathbb{R}$.

If $f(10) = 9$, then find $f(9)$

A. $y = x + h$ $\qquad \left|\dfrac{f(x+h) - f(x)}{h}\right| \leq |h|^2$

$\lim\limits_{h \to 0^+} -h^2 \leq \lim\limits_{h \to 0^+} (\sim) \leq \lim\limits_{h \to 0^+} h^2$

$\Rightarrow \lim\limits_{h \to 0^+} (\sim) = 0 \Rightarrow \underline{f'(x) = 0}$ $\forall x \in \mathbb{R}$

$\Rightarrow f(x) = c.$ Since $f(10) = 9$

$\Rightarrow \underline{f(x) = 9} \Rightarrow \underline{f(9) = 9}$

Q. If $|f(x) - f(y)| < (x-y)^2, \quad \forall x,y \in \mathbb{R}$

If $f(10) = 9$, then find $f(9)$

A. $|f(x) - f(y)| < |x-y|^2$

$y = (x+h)$ $\qquad \left|\dfrac{f(x+h) - f(x)}{h}\right| < |h|^2$

$\lim\limits_{h \to 0^+} -h < \lim\limits_{h \to 0^+} (\sim) < \lim\limits_{h \to 0^+} h$

By Sandwich Theorem, $\lim\limits_{h \to 0^+} (\sim) = 0$

$\Rightarrow \underline{f'(x) = 0}$ $\forall x \in \mathbb{R}$

$\Rightarrow f(x) = c$

Since $f(10) = 9 \Rightarrow \underline{f(x) = 9} \Rightarrow \underline{f(9) = 9}$

Q. Suppose $P(x) = a_0 + a_1 x + \cdots + a_n x^n$

If $|P(x)| \leq |e^{(x-1)} - 1|$ $\forall\ x \geq 0$, then

prove that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$

A. $|P(1)| \leq 0 \Rightarrow P(1) = 0$

$$\Rightarrow \left|\frac{P(x) - P(1)}{x-1}\right| \leq \left|\frac{e^{(x-1)} - 1}{(x-1)}\right|$$

$$\Rightarrow \lim_{x \to 1} (\sim) \leq \lim_{x \to 1} (\sim)$$

$$\Rightarrow |P'(1)| \leq 1 \Rightarrow |a_1 + 2a_2 + \cdots + na_n| \leq 1$$

Q. Let f be a fn^n s.t.

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right), \quad \forall\ x, y \in \mathbb{R},\ xy \neq 1$$

and $\lim_{x \to 0} \frac{f(x)}{x} = 2$

Find $f(1/\sqrt{3})$ & $f'(1)$

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A. $f(x) + f(0) = f(x) \Rightarrow f(0) = 0$

$f(x) + f(-x) = f(0) \Rightarrow f(-x) = -f(-x)$

Not req.

X $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 2 \Rightarrow f'(0) = 2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{f(x+h) + f(-x)}{h}$$

$$= \frac{f\left(\frac{x+h+(-x)}{1+(x+h)x}\right)}{h}$$

$$= \frac{f\left(\frac{h}{1+hx+x^2}\right)}{h} \quad \left(\frac{1}{\cancel{h}}\right)\left(\frac{\cancel{h}}{1+hx+x^2}\right)$$

2

$\Rightarrow f'(x) = \frac{2}{1+x^2} \Rightarrow f(x) = 2\tan^{-1}(x)$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$ $\quad$ $f'(1) = 1$

Q. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$ and

$g(x) = f(x-1) + f(x+1) \quad \forall x \in \mathbb{R}.$

Determine $g(x)$ in terms of 'x' & discuss its. cont. & diff.

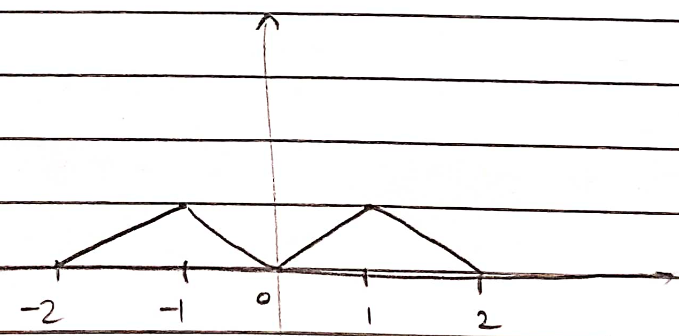
A.

$$f(x) = \begin{cases} 0 & , \ x < -1 \\ 1+x & , \ x \in [-1, 0) \\ 1-x & , \ x \in [0, 1] \\ 0 & , \ x > 1 \end{cases}$$

$$f(x-1) = \begin{cases} 0 & , \ x < 0 \\ x & , \ x \in [0, 1) \\ 2-x & , \ x \in [1, 2] \\ 0 & , \ x > 2 \end{cases}$$

$$f(x+1) = \begin{cases} 0 & , \ x < -2 \\ 2+x & , \ x \in [-2, -1) \\ -x & , \ x \in [-1, 0] \\ 0 & , \ x > 0 \end{cases}$$

$$g(x) = \begin{cases} 0 & , \ x < -2 \\ 2+x & , \ x \in [-2, -1) \\ -x & , \ x \in [-1, 0] \\ x & , \ x \in [0, 1) \\ 2-x & , \ x \in [1, 2] \\ 0 & , \ x > 2 \end{cases}$$



Q. Discuss continuity & diff. of $y = f(x)$ defined as

$$x = 2t - |t-1|$$
$$y = 2t^2 + t|t| \quad \forall \; t \in \mathbb{R}.$$

A. $t < 0$ $\quad x = 3t - 1 \quad \Rightarrow \quad y = \frac{(x+1)^2}{9}$

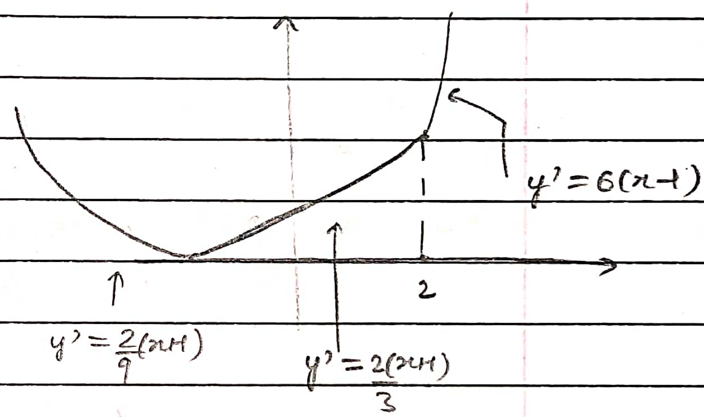
$\Rightarrow \underline{x < -1}$ $\quad y = t^2$

$\underline{t \in [0, 1)}$ $\quad x = 3t - 1 \quad \Rightarrow \quad y = \frac{(x+1)^2}{3}$

$\Rightarrow x \in [-1, 2)$ $\quad y = 3t^2$

$\underline{t \geq 1}$ $\quad x = t + 1 \quad \Rightarrow \quad y = 3(x-1)^2$

$\Rightarrow \underline{x \geq 2}$ $\quad y = 3t^2$



Q. Evaluate

(i) $\lim\limits_{x \to \infty} [\, l(x) - l(x+2) \,]$

A. $\lim\limits_{x \to \infty} \left[l\left(\frac{x}{x+2} \right) \right] = \left[l\left(\frac{1}{1 + \frac{2}{x}} \right) \right] = \underline{-1}$

(ii) $\lim\limits_{x \to \infty} [\, l(x) - l(x-2) \,]$

A. $\lim\limits_{x \to \infty} \left[l\left(\frac{x}{x-2} \right) \right] = \left[l\left(\frac{1}{1 - \frac{2}{x}} \right) \right] = 0$